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THE FIXED-POINT METHOD PRESENTS CERTAIN CHALLENGES THAT ARE IMPORTANT TO CONSIDER

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Abstract: The article reviews its origins, as well as the challenges that can arise in its use. Beyond its application in root calculus, the article explores its uses in integro differential and integral equations of the Vito Volterra and Hermann Von Helmholtz type, demonstrating the versatility of the method in various scientific problems. Hahn-Banach generalized this method, proving that every continuous and contractive function has a unique fixed point. His proof is based on Augustin Louis Cauchy's theorem, which states that every Cauchy sequence in a complete metric space is convergent. Because of the high level of abstraction required, this theory is not usually explored in depth in introductory mathematics courses.
Keywords: fixed point, calculus, roots, Hahn-Banach, Vito Volterra , Hermann Von Helmholtz, integro differential equations.

INTRODUCTION

The first to propose the fixed point method was H. Poincaré in 1886. Years later metric space is used which was introduced by M. Frechet in 1906, which provided the development of a large number of mathematical contributions, physics and other areas of knowledge, in which appears the normed spaces or distance in Banach spaces where this is closed and convex. The first fixed point theorem is due to L.E.J. Brouwer in 1912. Fixed point theory has played an important role in the problems of nonlinear functional analysis, which is the combination of analysis, topology and algebra. In 1967, B.N. Sadovski gives a fixed point theorem which states that, if E is a Banach space, X is a closed, bounded and convex subset of the space E, A mapping of a domain then has a fixed point in X. In 1968 Browder stated the following theorem, Let K be a nonempty compact convex subset of a topological vector space. Let T be a map of K into 2^K, where for each K, T (x) is a nonempty convex subset of K. Assume further that for every y in K,

$T_1(y) = \{x \in K, : y \in T(x)\}$ Then there exists x_0 in K such that $x_0 \in T(x_0)$ [1-2].

One of the applications is in the Fredholm integro-differential equation given by $x(t) = \int_a^b f(t,s,x(s),x'(s))ds + g(t),\ t\epsilon[a,b]$ where $f: [a,b]x[a,b]XxX \to X$ is continuous, X is a Banach space and $g \in C^1([a,b],X)$ to find the fixed point, this equation is taken to a system of equations derived with respect to t which reduce to the following system of integrable equations, $x(t) = \int_a^b f(t,s,x(s),y(s))ds + g(t), \int_a^b \frac{\partial}{\partial t} f(t,s,x(s),y(s))ds + g'(t),\ t\epsilon[a,b]$ which is a slightly more complex form of the method.

For our case we will only deal with the calculation of a simple root, for this it is necessary that it is a continuous function, of which we want to write in the form f: $\mathbb{R} \to \mathbb{R}$ a continuous function, which we want to write in the form:

$$x = G(x) \qquad (1)$$

of equation (1) represents the curve as $y=G(x)$ and the straight line identity y=x. where their intersection corresponds to the fixed points. The projection on the horizontal axis ("x"), are the roots of the equation according to figure 1.

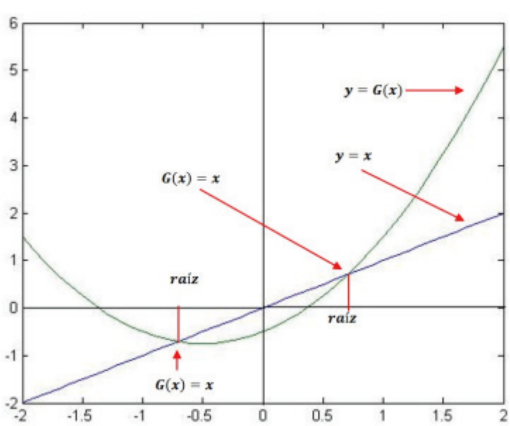

Figure 1: Geometric interpretation of the intersection of the line and the cave.

From Equation (1) it must be fulfilled that

$$f(x) = x - G(x) \qquad (2)$$

An arbitrary x_0 arbitrary thus the iterative form of Eq. (1) is obtained Eq. (3)

$$x_{n+1} = G(x_n) \qquad (3)$$

From Eq. (3) we generate a sequence of points x_0, x_1, ... x_n

The stop criterion is

$$|x_{n+1} - x_n| \leq \varepsilon \qquad (4)$$

where is the error of the approximation.

CONVERGENCE ORDER

A sufficient condition for convergence is easily expressed in terms of the first derivative of g(x) and using the mean value theorem of elementary calculus, which states that for some z between a

$$g(b) - g(a) = g'(z)(b - a) \quad (5)$$

Now consider the expression $|x_{n-1}-x_n|$ where the values of x_i are repeated in the iterative formula of Eq. (6) :

$$x_{n-1} = g(x_n) \; (6)$$

We have, by the mean value theorem, for some z between $x_{n-1}-x_n$

$$|x_{n+1} - x_n| = |g(x_n) - g(x_{n-1})| =$$

$$|g'(z)(x_{n+1} - x_n)| \quad (7)$$

A demonstration can be found at[3] . Suppose that g' is bounded in magnitude by some number $M \geq 0$ clearly, if there is a constant $M_0 = sup\left\{\frac{|f(x)-f(y)|}{|x-y|} \text{ for } x, y \in [a,b], x \neq y\right\}$ and it is said that M_0 is the Lipschitz constant of f a sufficiently large interval contains all the x_i Then the following development is obtained

$$|x_{n+1} - x_n| \leq M|x_n - x_{n-1}| =$$

$$M|g(x_{n-1}) - g(x_{n-2})| \; (8)$$

$$\leq M^2|x_{n-1} - x_{n-2}| \qquad (9)$$

$$= M^2|g(x_{n-2}) - g(x_{n-3})| \; (10)$$

$$\leq M^2|x_{n-2} - x_{n-3}| \qquad (11)$$

Until.

$$\leq M^n|x_{n-2} - x_{n-3}| \qquad (12)$$

A sufficient condition for convergence is therefore $M<1$ in the literature is used, $|g'(x)| \leq 1$ where g it must be continuous and its derivative must also be continuous for $x \in I$ where I is an open interval, where α at I which is the root of f mathematically $f(\alpha)=0$ in terms of approximation we have in general that $|g'(x_0) - g'(x_1)| \leq L|x_0-x_1|$ L is called Lispschitz constant if one has $M_0 \leq M$ then we have:

$$|x_{n+1} - x_n| \leq M^n|x_{n-2} - x_{n-3}| \qquad (13)$$

$$\frac{|x_{n+1}-x_n|}{|x_{n-2}-x_{n-3}|} = \left|\frac{x_{n+1}-x_n}{x_{n-2}-x_{n-3}}\right| \leq M^n,$$

$$\left|\frac{x_{n+1}-x_n}{x_{n-2}-x_{n-3}}\right| \leq M^n \qquad (14)$$

That it is shown that the order of linear convergence if $-1 \leq \boldsymbol{M} \leq \boldsymbol{1}$

METHODOLOGY OR DEVELOPMENT

The method has a disadvantage inherent in the diversity of forms that equation (1) can take. This diversity complicates the choice of a suitable method and thus affects convergence. Although finding a suitable form of the equation can significantly accelerate convergence, it does not guarantee its success. Analogous to the compactness and completeness criteria in metric spaces, where the Cauchy condition is necessary but not sufficient for convergence, in the considered workspace, the required condition is also necessary but not sufficient, as will be shown below.

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APPLICATION1

Let the equation be.

$$f(x) = x^2 - 2 = 0 \quad (15)$$

Through some algebraic manipulations, it is possible to obtain the following G(x):

$$x = x^2 + x - 2 \text{ where } G(x) = x^2 + x - 2 \quad (16)$$

$$x = \frac{2}{x} \text{ where } G(x) = \frac{2}{x} \quad (17)$$

$$x = \pm\sqrt{x} \quad \text{where } G(x) = \pm\sqrt{x} \quad (18)$$

$$x = \frac{\left(x+\frac{2}{x}\right)}{2} \text{ where } G(x) = \frac{\left(x+\frac{2}{x}\right)}{2} \quad (19)$$

So any continuous function $f(x)$ be it algebraic, transcendental, exponential, or combinations of these, can have quite a few continuous functions, this is one of the problems that complicates its application. $G(x)$ This is one of the problems that complicates its application, the other is the application of the convergence criterion of the method. Then from equation (15) one finds $G(x)$:

$$y = x \text{ where } y = G(x) = \frac{\left(x+\frac{2}{x}\right)}{2} \quad (20)$$

From equation (20). It is observed that they have two fixed points shown in figure (1).

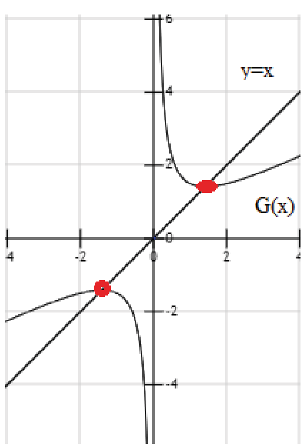

Figure 1. The red dots represent the fixed points.

$$y = x \text{ where } y = G(x) = x^2 + x - 2 \quad (21)$$

Figure (2) shows equation (21). is given by:

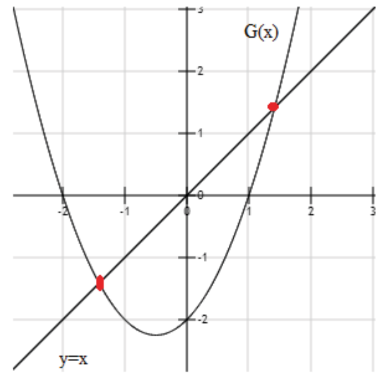

Figure (2). The fixed points are shown.

$$y = x \text{ where } y = G(x) = \frac{2}{x} \quad (22)$$

Figure 3 shows the two fixed points of equation (22).

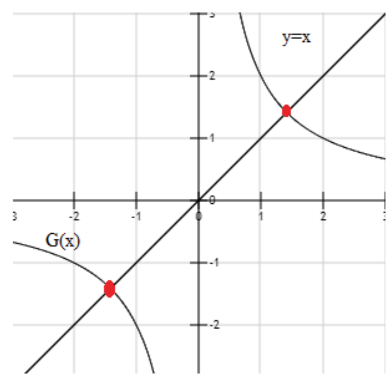

Figure 3. Shows the two fixed points

Writing equation (19) in its iterative form. $x_{n+1} = \frac{\left(x_n+\frac{2}{x_n}\right)}{2}$ with initial value initial value $x_0=1.2$ approximation error 0.05 (stopping criterion)

Iteration Number	Approximate value
1	1.433333
2	1.41424

Table 1. Values obtained using the iterative form of the equation (19).

APPLICATION 2

Consider the nonlinear equation $x^3=2x+1$ then:

$$x = \frac{x^3-1}{2}\ ,\ G(x) = \frac{x^3-1}{2} \quad (23)$$

$$x = \frac{1}{x^2-2}\ ,G(x) = \frac{1}{x^2-2} \quad (24)$$

$$x = \sqrt{\frac{2x+1}{x}}\ ,G(x) = \sqrt{\frac{2x+1}{x}} \quad (25)$$

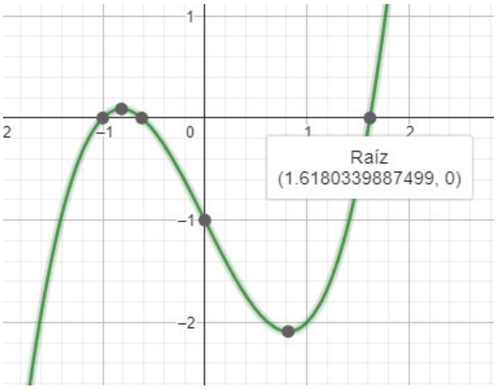

Figure (4) shows the roots of the equation $x^3=2x+1$

Figure (5). shows the function $G(x)$ where it shows the intersection with the identity function which is the fixed point.

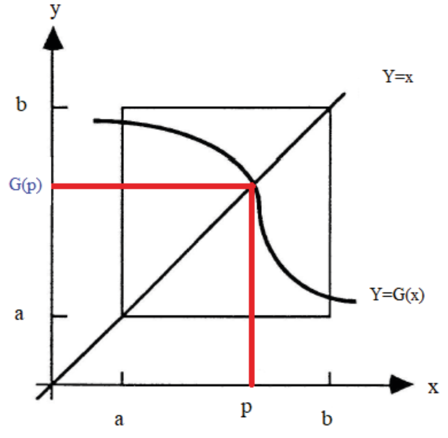

Figure (5). It is shown that $G(p)$ has a single fixed point at $pe(a,b)$

THEOREM 1

If $G\in C^1[a,b]$ y $G(x)[a,b]$ for all $x\in[a,b]$ then G has a fixed point at $[a,b]$ if, in addition, $G'(x)$ exists and is continuous at (a,b) then G(x) has a single point at p at (a,b) y $G'(x) \leq k<1$, for all $x\in(a,b)$

Demonstration:

If $G(a)= a$ o $G(b)= b$ the existence of the fixed point is obvious. Suppose that it is not, then it must be satisfied that $G(a)>a$ y $G(b)<b$. Let us define $h(x)=G(x)-x$, $h(x)$ continuous in $[a,b]$ y $h(a)=G(a)-a>0$, $h(b)=G(b)-b>0$ using the intermediate value theorem implies that there exists an $p\in(a,b)$ such that $h(p)=0$ therefore $G(p)-p=0$, o $G(p)=p$ y p is a fixed point of G.

THEOREM 2

Let $G(x)$ a continuous function with a continuous derivative the method converges if

$$|G'(x_0)| < 1\ para\ x\epsilon\mathbb{R} \quad (26)$$

this is a necessary but not sufficient condition, in equation (9) its derivative is not continuous either. If we take $x_0=1$ the method calculates a root.

Performing an analysis of Eq. (23) if $<G'(x)<1$ has asymptotic convergence, and if $1<G'(x)<0$ is oscillatory convergence for a $x\in\mathbb{R}$. Otherwise, the method diverges [3].

If p is a fixed point it satisfies that $f(p)=0$.

The following are the graphs of the possible cases of the $g'(x)$ as well as how they behave when the iterative method is performed.

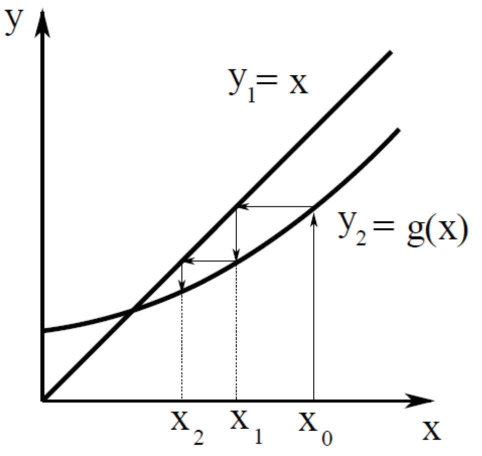

Figure (6). Convergence in $0<g'(x)<1$ monotonic behavior [4].

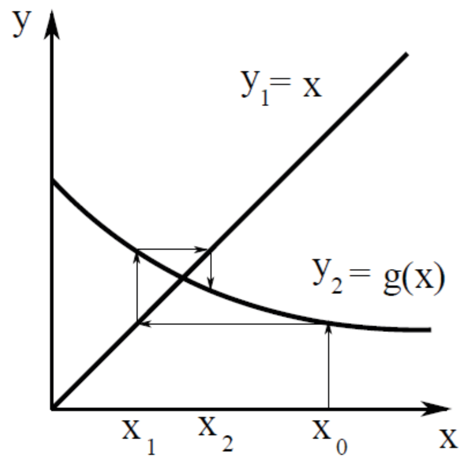

Figure (7). Convergence in $-1<g'(x)<0$ oscillatory behavior [4].

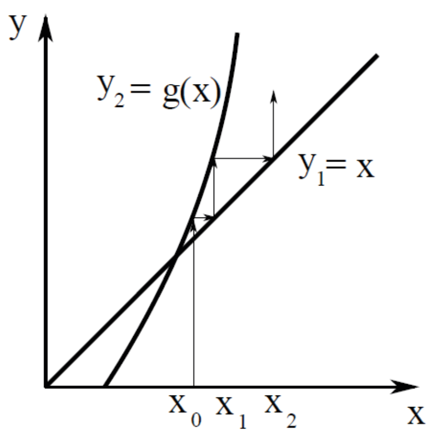

Figure (8). Even divergence $<g'(x)<1$ (monotonic behavior) [4].

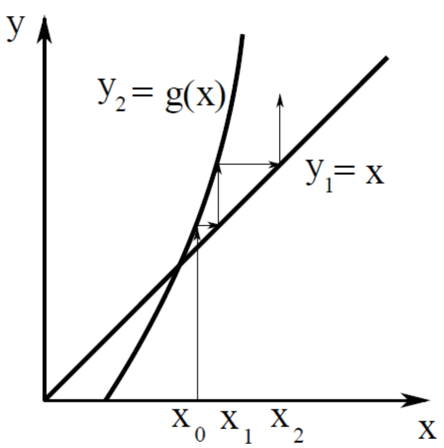

Figure (9). Divergence $g'(x)<-1$ (oscillatory behavior) [4].

By applying the method in a generalized form, using the function defined in equations (23-25) as the basis for constructing a new iterative function $G(x)$ defined in equations (23-25) as a basis for constructing a new iterative function, the following iterative forms are obtained.

$$x_{n+1} = \frac{x_n^3-1}{2} \quad (27)$$

$$x_{n+1} = \frac{1}{x_n^2-2} \quad (28)$$

$$x_{n+1} = \sqrt{\frac{2x_n+1}{x_n}} \quad (29)$$

$$x_{n+1} = \sqrt{\frac{2x_n+5}{2x_n+4}} \quad x \neq -2 \quad (30)$$

Using Equation (27), the following results are obtained: initial value 0.5, approximation error 0.05 (stopping criterion).

Iteration number	approach to the root
1	1.187500
2	0.337280
3	-0.480816
4	-0.555578
5	-0.585744

Table 2. Values obtained using the iterative form of the equation (27)

By applying Equation (28), it is possible to obtain the following results: initial value 0.5, approximation error 0.05 (stopping criterion).

Iteration number	Approximation to the root
1	4.000000
2	0.71429
3	-0.501279
4	-0.571847
5	-0.597732

Table 3. Values obtained using the iterative form of the equation (28)

By applying Equation (29), it is possible to obtain the following results: initial value 0.5, approximation error 0.05 (stopping criterion).

Iteration number	Approximation to the root
1	1.632993
2	1.616284

Table 4. Values obtained using the iterative form of the equation (29)

Using Equation (30), the following results are obtained: initial value 0.5, approximation error 0.05 (stopping criterion).

Iteration number	Approximation to the root
1	1.069045
2	1.078386

Table 5. Values obtained using the iterative form of the equation (30)

RESULTS AND ANALYSIS

The method converges rapidly if a suitable initial approximation is found. Although it is not strictly necessary that the function and its derivative be continuous over the entire domain, as in the case of equation (20), to guarantee convergence it is usually required that the derivative of the function be continuous in an environment of the root. This condition, although not sufficient, limits the applicability of the method. As can be seen in Figure (7), even starting from a value very close to the fixed point, the method may diverge or enter a cycle. Therefore, the convergence of the method depends largely on the choice of the initial approximation and the properties of the function in a root environment.

CONCLUSIONS

When analyzing equations (20), (29), (17), (27) and (28), it is observed that not all of them satisfy the established convergence criterion. Equations (29) and (30), not being continuous at 0, -2, converge to an approximation of the root, while equations (17), (27) and (28) exhibit oscillatory or divergent behavior. These results show that the continuity condition, although necessary, is not sufficient to guarantee the convergence of the method. Numerical experiments performed in application 2 corroborate these observations, showing that some sequences approach the root without reaching it. The examples presented illustrate the complexity and limitations of the method when applied to problems with different characteristics.

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