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CONTRACTION SEMIGROUP IN SPACE $L^2([-\pi, \pi])$

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Abstract: In this work we begin studying the differential operator: H_o *in the space*: $L^2([\pi, \pi])$. We know that this operator is not bounded, it is densely defined and symmetric and therefore does not admit a symmetric linear extension to the entire space. We introduce a family of operators in the space: $L^2([\pi, \pi])$ and we show that this forms a class contraction semigroup:C_o, and it has H_o as its infinitesimal generator. We also prove that if we restrict the domains of that family of operators, they still remain a contraction semigroup.
Finally, we give results of the existence of a solution to the associated abstract Cauchy problem and properties of continuous dependence of the solution in connection with other norms.
Keywords: Space: $L^2([\pi, \pi])$, Hellinger-Toeplitz theorem, Parseval identity, contraction semigroup, existence of solution, norm of the graph.

INTRODUCTION

In this article we will study some operators in the space $L^2([-\pi, \pi])$. That is, we will introduce the differential operator that is not bounded and we will prove that it is bounded with the norm of the graph. We will introduce a family of operators in $L^2([-\pi, \pi])$ and we will show that they are bounded and that they form a class contraction semigroup: C_o, having the differential operator as an infinitesimal generator. Now, restricting the domain of this family of operators, we will prove that it continues to form a contraction semigroup of class: C_o. Thus, we will improve the solution existence result for the associated abstract Cauchy problem. We can cite some references for the treatment of solution existence via semigroups, for example [1], [3], [4], [5] and [6].

Our article is organized as follows. In section 2, we indicate the methodology used and cite the references used. In section 3, we present the results obtained from our study. We divide this section into seven subsections. Thus, in subsection 3.1 we quickly study the Differential operator in $L^2([-\pi, \pi])$. In subsection 3.2, we prove that the introduced family of operators forms a class contraction semigroup:

C_o in $L^2([-\pi, \pi])$. In subsection 3.3,

we calculated the infinitesimal generator of the C_o - contraction semigroup and we have the first result of the existence of a solution for the associated abstract Cauchy problem and also the continuous dependence of the solution on the initial data. In subsection 3.4, we introduce the norm of the graph in the Ho domain that makes it a Hilbert space and prove that H_o is limited with this norm. In subsection 3.5, we introduce other norms equivalent to the graph norm. In subsection 3.6, we prove that the family of operators with restricted domain continues to be a contraction semigroup. In subsection 3.7, we have the result of existence of solution in connection with other standards.

Finally, in section 4 we give the conclusions and observations of this study.

METHODOLOGY

We will quickly introduce some definitions that will be used in this article.

Definition 2.1: we have: P the space of the infinitely different functions f: IR → $\mathbb{C}$ diferenciables and periodic with period 2π. This space is also denoted by $C^{\infty}_{per}([-\pi,\pi])$.

Asse Test that P is a complete metric space.

$$\begin{aligned} P' \;&:=\; \Big\{T : P \longrightarrow \mathbb{C} \text{ lineal tal que } \exists \psi_n \in P \text{ y} \\ &\qquad < T, \varphi >= \lim_{n\to+\infty} \int_{-\pi}^{\pi} \psi_n(x)\varphi(x)\,dx\,,\ \forall \varphi \in P \Big\} \\ &=\; (P)'\,. \end{aligned}$$

That is, P ‘ is the topological dual of P. Thus, P ‘ is called the space of the Periodic

Distributions.

Definition 2.2 We defined space:

$L^2(-\pi, \pi]) := \{f \in P', \exists\ \varphi n$ *Cauchy sequence in* $\|\cdot\|_2\ y\ \varphi_n \xrightarrow{P'} f\} \subset P'$

It is proven that L2([−π, π]) is a $\mathbb{C}$ - Hilbert space.

To view properties of *P*, *P*ˋ y $L^2([-\pi, \pi])$ we quoted [1], [7] and [8]; and for the theory of semigroups, we cite [3] and [4].

Now, we will state an important result that will be used later.

Theorem 2.1 (Hellinger-Toeplitz) If T is an unbounded, symmetric and densely defined linear operator (for example: $(i.e.\ \overline{Dom(T)} = H)$ on a Hilbert space *H*, therefore, it does not admit a symmetric linear extension to *H*.

Test: We cited Kreyszig [2].

MAIN RESULTS

THE DIFFERENTIAL OPERATOR: H_o IN $L^2([-\pi, \pi])$

We will introduce the following application

Definition 3.1 (Differential Operator: H_o). Let's define the application

$$\begin{array}{rcl} H_o : Dom\,(H_o) \subset L^2([-\pi,\pi]) & \longrightarrow & L^2([-\pi,\pi]) \\ f & \longrightarrow & H_o f := -f'' \end{array}$$

distributional derivative where $Dom(H_o) := \{f \in L^2([-\pi, \pi])$ *tal que* $-f'' \in L^2([-\pi, \pi])\}$.

H_o is known as Differential Operator. We will remember its properties with the following proposition.

Note 3.1: Due to the Fourier Transform, we have to $H_o f = (k^2 f(k))^v$

for every $f \in Dom(H_o) = \{f \in L([-\pi, \pi])$ *and* $(k f(k)) \in l(Z)\}$.

Proposition 3.1 The Differential operator H_o is $\mathbb{C}$ - linear, densely defined, symmetrical and unbounded. Besides, H_o does not support symmetric linear extension to $L^2([-\pi, \pi])$.

Test: The Proof can be seen in [9], where Hellinger-Toeplitz Theorem 2.1 is used.

SEMI GROUP OF CLASS C_O IN $L^2([-\pi, \pi])$

Proposition 3.2 (Semi group of class: C_o) So, we have $t \geq 0$, we defined the applications

$$e^{-tH_o} f = (e^{-tk^2}\widehat{f}(k))^\vee,\ \forall f \in L^2([-\pi,\pi])$$

therefore,

$$\{e^{-tH_o}\}_{t\geq 0} \subset B(L^2([-\pi,\pi]))\ \ y$$

It also forms a class contraction semigroup C_o en $L^2([-\pi, \pi])$.

Test: En $t = 0$, sea $f \in L^2([-\pi, \pi])$ we have $e^{-0H}o\, f = (e^{-0k}2\, f(k))^v = (f(k))^v = f$, therefore:

$$e^{-0H}o = I, \tag{3.1}$$

where I is the identity operator on $L^2([-\pi, \pi])$.

Now we will prove that $\{e^{-tH}o\}_{t\geq 0}$ is a family of bounded linear operators and of contraction, for example:

$$\|e^{-tH_o}\| \leq 1,\ \forall t \geq 0.$$

In fact, we have: $t > 0$ y $f \in L^2([-\pi, \pi])$,

$$\begin{aligned} \||e^{-tH_o} f\||_2^2 &= 2\pi \sum_{k=-\infty}^{+\infty} |e^{-tk^2}\widehat{f}(k)|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} |e^{-tk^2}|^2 |\widehat{f}(k)|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} \underbrace{e^{-2tk^2}}_{\leq 1} |\widehat{f}(k)|^2 \\ &\leq 2\pi \sum_{k=-\infty}^{+\infty} |\widehat{f}(k)|^2 \\ &= \||f\||_2^2 < \infty\,. \end{aligned} \tag{3.2}$$

Therefore, of (3.2) we have that $e^{-tHo} f \in L^2([-\pi, \pi])$, it means, e^{-tHo} it is well defined for $t \geq 0$. On the other hand, it is evident that: e^{-tHo} is $\mathbb{C}$ - linear:

$$\begin{aligned} e^{-tH_o}(f+cg) &= \left(e^{-tk^2}\widehat{f+cg}(k)\right)^\vee \\ &= \left(e^{-tk^2}\{\widehat{f}(k)+c\widehat{g}(k)\}\right)^\vee \\ &= \left(e^{-tk^2}\widehat{f}(k)+ce^{-tk^2}\widehat{g}(k)\right)^\vee \\ &= \left(e^{-tk^2}\widehat{f}(k)\right)^\vee + c\left(e^{-tk^2}\widehat{g}(k)\right)^\vee \\ &= e^{-tH_o}f + c\,e^{-tH_o}g\,, \end{aligned}$$

So, for $f, g \in L^2([-\pi, \pi])$ and $c \in \mathbb{C}$

Thus, from (3.2) we also have to $|||e^{-tH_o}f|||_2 \leq |||f|||_2,\ \forall f \in L^2([-\pi,\pi])$. That is, the operator: e^{-tH_o} is limited and

$$\|e^{-tH_o}\| \leq 1\,,\ \forall t \geq 0\,. \tag{3.3}$$

So, we have: $t > 0$, $r > 0$ and $f \in L^2([-\pi, \pi])$, we have

$$\begin{aligned} e^{-(t+r)H_o}f &= \left(e^{-(t+r)k^2}\widehat{f}(k)\right)^\vee \\ &= \left(e^{-tk^2}e^{-rk^2}\widehat{f}(k)\right)^\vee \\ &= \left(e^{-tk^2}\{e^{-rH_o}f\}^\wedge(k)\right)^\vee \\ &= e^{-tH_o}\{e^{-rH_o}f\} \\ &= e^{-tH_o}\circ e^{-rH_o}f \end{aligned}$$

It means, $e^{-(t+r)H_o} = e^{-tH_o} \circ e^{-rH_o}$ for $t > 0$ and $r > 0$. The case: $t = 0$ o $r = 0$, it is evident; therefore

$$e^{-(t+r)H_o} = e^{-tH_o} \circ e^{-rH_o},\ \forall t, r \geq 0. \tag{3.4}$$

So, we have $f \in L^2([-\pi, \pi])$, we will prove: $|||e^{-tH_o}f - f|||_2 \longrightarrow 0$ when $t \to 0^+$. In fact,

$$\begin{aligned} &|||e^{-tH_o}f - f|||_2^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} |e^{-tk^2}\widehat{f}(k) - \widehat{f}(k)|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} |(e^{-tk^2}-1)\widehat{f}(k)|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} \underbrace{|e^{-tk^2}-1|^2}_{M(k,t):=}|\widehat{f}(k)|^2 \end{aligned} \tag{3.5}$$

Where: $\lim_{t\to 0+} M(k, t) = 0$.

Furthermore, the kth term of the series (3.5) is factored:

$$M(k,t)|\widehat{f}(k)|^2 \leq 4|\widehat{f}(k)|^2$$

and like the series $\sum_{k=-\infty}^{+\infty} |\widehat{f}(k)|^2$ is convergent, thus, using the Weierstrass M-Test we have that the series converges absolutely and uniformly.Therefore,

$$\lim_{t\to 0^+} |||e^{-tH_o}f - f|||_2^2 = 2\pi \sum_{k=-\infty}^{+\infty} \underbrace{\lim_{t\to 0^+} |e^{-tk^2}-1|^2}_{=0}|\widehat{f}(k)|^2 = 0\,.$$

Thus, we have proven that:

$$\lim_{t\to 0^+} |||e^{-tH_o}f - f|||_2 = 0\,,\ \forall f \in L^2([-\pi,\pi])\,. \tag{3.6}$$

Of (3.1), (3.4), (3.3) y (3.6), we concluded that $\{e^{-tH_o}\}_{t\geq 0}$ is a semigroup of contraction of class: C_o en $L^2([-\pi, \pi])$.

Proposition 3.3 $\forall f \in L^2([-\pi, \pi])$, the application: $t \to e^{-tH_o} f$ is continuous $[0, \infty)$ *to* $L^2([-\pi, \pi])$.

Test: From (3.6) we have the continuity at 0 to the right. Thus, we focus on testing continuity at $t > 0$.

Let $h > 0$, using the semigroup property, the inequality (3.3) and the limit (3.6), we have:

$$\begin{aligned} &|||e^{-(t+h)H_o}f - e^{-tH_o}f|||_2 \\ &= |||e^{-tH_o}e^{-hH_o}f - e^{-tH_o}f|||_2 \\ &= |||e^{-tH_o}\{e^{-hH_o}f - f\}|||_2 \\ &\leq |||e^{-hH_o}f - f|||_2 \ \to\ 0 \end{aligned} \tag{3.7}$$

when $h \to 0^+$.

Now, considering $h > 0$ such that $t - h > 0$ and proceeding analogously as in (3.7), we have:

$$\begin{aligned} &|||e^{-(t-h)H_o}f - e^{-tH_o}f|||_2 \\ &= |||e^{-(t-h)H_o}f - e^{-(t-h)H_o}e^{-hH_o}f|||_2 \\ &= |||e^{-(t-h)H_o}\{f - e^{-hH_o}f\}|||_2 \\ &\leq |||e^{-hH_o}f - f|||_2 \ \to\ 0 \end{aligned} \tag{3.8}$$

when $h \to 0^+$.

From (3.7) and (3.8) we have that the application is continuous in $t \in \mathbb{R}^+$.

Proposition 3.4 If $Si\ f_n \overset{|||\cdot|||_2}{\longrightarrow}$ *therefore,* $|||e^{-tH_o}f_n - e^{-tH_o}f|||_2 \to 0$ $n \to +\infty$.

Test: It is immediate since from (3.2) we have:

$$|||e^{-tH_o}f_n - e^{-tH_o}f|||_2 =$$
$$|||e^{-tH_o}(f_n - f)|||_2 \leq |||f_n - f|||_2 .$$

G.I. CALCULATION OF $\{e^{-tHo}\}_{t\geq 0}$ en $L^2([-\pi, \pi])$

Proposition 3.5 *The −Ho* operator is the infinitesimal Generator (G.I.) of the contraction semi-group: $\{e^{-tHo}\}_{t\geq 0}$ en $L^2([-\pi, \pi])$.

Test: If A is the G.I. of the contraction semigroup $\{e^{-tHo}\}_{t\geq 0}$ en $L^2([-\pi, \pi])$ therefore, It all comes down to proving that $Dom(A) = Dom(H_o)$ y $A = -H_o$.

1. $Dom(H_o) \subset Dom(A)$: In fact, we have $f \in Dom(H_o)$ therefore, $H_o f := (k^2\hat{f}(k))^{\vee}$, where $f \in L^2([-\pi, \pi])$ y $(k^2\hat{f}(k)) \in l^2(\mathbb{Z})$, for example:

$$\sum_{k=-\infty}^{+\infty} |k^2\widehat{f}(k)|^2 < \infty \tag{3.9}$$

For $t > 0$, we have

$$\left|\left|\left|\frac{e^{-tH_o}f - f}{t} + H_o f\right|\right|\right|_2^2$$
$$= 2\pi \sum_{k=-\infty}^{+\infty} \left|\frac{e^{-tk^2}\widehat{f}(k) - \widehat{f}(k)}{t} + k^2\widehat{f}(k)\right|^2$$
$$= 2\pi \sum_{k=-\infty}^{+\infty} \left|\left\{\underbrace{\frac{e^{-tk^2} - 1}{t} + k^2}_{H(k,t):=}\right\}\widehat{f}(k)\right|^2$$

where $\lim_{t\to 0} H(k, t) = 0$. Besides, we have:

$$|H(k,t)|^2|\widehat{f}(k)|^2 \leq 4k^4|\widehat{f}(k)|^2$$

and since (3.9), using the Weierstrass M-test we have that the series $\sum_{k=-\infty}^{+\infty} |H(k,t)|^2|\widehat{f}(k)|^2$ converges absolutely and uniformly,therefore:

$$\lim_{t\to 0+}\left|\left|\left|\frac{e^{-tH_o}f - f}{t} + H_o f\right|\right|\right|_2^2 =$$
$$2\pi \sum_{k=-\infty}^{+\infty} \underbrace{|\lim_{t\to 0+} H(k,t)|^2}_{=0}|\widehat{f}(k)|^2 = 0 .$$

So: $\left|\left|\left|\frac{e^{-tH_o}f-f}{t} + H_o f\right|\right|\right|_2 = 0.$ That is, there exists $\lim \left\{\frac{e^{-tH_o}f-f}{t}\right\} = -H_o f.$ Therefore, $f \in D(A)$ y $Af = -H_o f$.

2. $Dom(An) \subset Dom(,H_o)$. : So, we have $f \in Dom(A)$ therefore, $f \in L^2([-\pi, \pi])$ y $\lim \left\{\frac{e^{-tH_o}f-f}{t}\right\} = Af$ $L^2([-\pi, \pi])$. So, we have:

$$\lim_{t\to 0+}\left|\left|\left|\frac{e^{-tH_o}f - f}{t} - Af\right|\right|\right|_2 = 0 .$$

So, we have: $\in > 0$

$$\epsilon > \frac{1}{2\pi}\left|\left|\left|\frac{e^{-tH_o}f - f}{t} - Af\right|\right|\right|_2^2$$
$$= \sum_{k=-\infty}^{+\infty}\left|\frac{e^{-tk^2}\widehat{f}(k) - \widehat{f}(k)}{t} - \{Af\}^\wedge(k)\right|^2$$
$$> \left|\frac{e^{-tk^2}\widehat{f}(k) - \widehat{f}(k)}{t} - \{Af\}^\wedge(k)\right|^2, \ \forall k \in \mathbb{Z} .$$

Then, for each $k \in Z$,

$$\frac{e^{-tk^2}\widehat{f}(k) - \widehat{f}(k)}{t} \to \{Af\}^\wedge(k)$$

when $t \to 0^+$,
but we know that

$$\frac{e^{-tk^2}\widehat{f}(k) - \widehat{f}(k)}{t} \to -k^2\widehat{f}(k)$$

when $t \to 0^+$,
For each $k \in Z$.
Then, for each: $k \in Z$, we have $\{Af\}^\wedge(k) = -k^2 f^\wedge(k)$. Therefore,

$$l^2(\mathbb{Z}) \ni \{Af\}^\wedge = (-k^2\widehat{f}(k)) . \tag{3.10}$$

From (3.10) we have to $(-k^2\widehat{f}(k)) \in l^2(\mathbb{Z})$, esto es $f \in Dom(H_o)$ y $Af = -H_o f$.

From the two items it is concluded that: $Dom(A) = Dom(H_o)$ and $A = -H_o$.

Proposition 3.6 So, we have: $t \geq 0$, *si* $f \in Dom(H_o)$ therefore, $e^{-tHo} f \in Dom(H_o)$. Furthermore, it is fulfilled: $H_o e^{-tHo} f = e^{-tHo} H_o f$, $\forall f \in Dom(H_o)$.

Test: In fact, we have: $f \in Dom(H_o)$, $t > 0$, $r > 0$ y $-H_o$ el G. I. of $\{e^{-tHo}\}_{t\geq 0}$ en $L^2([-\pi, \pi])$, therefore:

$$\begin{aligned} &\lim_{r\to 0^+} \left\{ \frac{e^{-rH_o}(e^{-tH_o} f) - e^{-tH_o} f}{r} \right\} \\ = &\lim_{r\to 0^+} \left\{ \frac{e^{-tH_o}(e^{-rH_o} f) - e^{-tH_o} f}{r} \right\} \\ = &\lim_{r\to 0^+} e^{-tH_o} \left\{ \frac{e^{-rH_o} f - f}{r} \right\} \\ = &\ e^{-tH_o} \left[\lim_{r\to 0^+} \left\{ \frac{e^{-rH_o} f - f}{r} \right\} \right] \\ = &\ e^{-tH_o}[-H_o f] \in L^2([-\pi, \pi]). \end{aligned}$$

Thus, there is a limit in $L^2([-\pi, \pi])$. It is: $e^{-tHo} f \in Dom(H_o)$ y

$$-H_o(e^{-tH_o} f) = e^{-tH_o}[-H_o f] = -e^{-tH_o}[H_o f],$$

i.e.

$$H_o \circ e^{-tH_o} f = e^{-tH_o} \circ H_o f\,,\ \forall f \in Dom(H_o)\,. \tag{3.11}$$

Proposition 3.7 *Si* $f \in Dom(H_o)$ therefore, the application*:* $t \to e^{-tHo} f$*, of* $(0, \infty)$ *a* $L^2([-\pi, \pi])$*, is* differentiable in $(0, \infty)$ and the derived $-e^{-tH}o\, H_o f$. Besides. $\frac{\partial}{\partial t}\{e^{-tHo} f\} = -H_o e^{-tHo} f$.

Besides $\frac{\partial}{\partial t}\{e^{-tH_o} f\} = -H_o e^{-tH_o} f$.

Test: So, we have: $t > 0$, $h > 0$ and $0 < t - h$, we have

$$\begin{aligned} &\frac{e^{-tH_o} f - e^{-(t-h)H_o} f}{h} + e^{-tH_o} H_o f \\ &= e^{-(t-h)H_o} \left\{ \frac{e^{-hH_o} f - f}{h} \right\} + e^{-tH_o} H_o f \\ &= e^{-(t-h)H_o} \left\{ \frac{e^{-hH_o} f - f}{h} \right\} \pm e^{-(t-h)H_o} H_o f + e^{-tH_o} H_o f \\ &= e^{-(t-h)H_o} \left\{ \frac{e^{-hH_o} f - f}{h} + H_o f \right\} - e^{-(t-h)H_o} H_o f + e^{-tH_o} H_o f\,. \end{aligned} \tag{3.12}$$

$\|e^{-(t-h)Ho}\| \leq 1$, $\frac{e^{-hH_o} f - f}{h} \longrightarrow -H_o f$ when $h \to 0^+$ and $e^{-(t-h)Ho} H_o f \to e^{-tHo} H_o f$ *when* $h \to 0^+$, taking limit to (3.12) when $h \to 0^+$ we have

$$\lim_{h\to 0^+} \left\{ \frac{e^{-tH_o} f - e^{-(t-h)H_o} f}{h} + e^{-tH_o} H_o f \right\} = 0\,,$$

this is

$$\lim_{h\to 0^+} \left\{ \frac{e^{-tH_o} f - e^{-(t-h)H_o} f}{h} \right\} = -e^{-tH_o} H_o f\,. \tag{3.13}$$

Similarly we proceed, when $h > 0$, it is,

$$\frac{e^{-(t+h)H_o} f - e^{-tH_o} f}{h} = e^{-tH_o} \left\{ \frac{e^{-hH_o} f - f}{h} \right\}. \tag{3.14}$$

As $e^{-tH}o \in B(L^2([-\pi, \pi])$ and $f \in Dom(-H_o)$, taking limit to (3.14) when $h \to 0^+$, we have

$$\begin{aligned} &\lim_{h\to 0^+} \left\{ \frac{e^{-(t+h)H_o} f - e^{-tH_o} f}{h} \right\} \\ &= e^{-tH_o} \left\{ \lim_{h\to 0^+} \left\{ \frac{e^{-hH_o} f - f}{h} \right\} \right\} \\ &= e^{-tH_o}\{-H_o f\} \\ &= -e^{-tH_o} H_o f\,. \end{aligned} \tag{3.15}$$

Of (3.13 and (3.15) we have that

$$\exists \underbrace{\lim_{h\to 0} \left\{ \frac{e^{-(t+h)H_o} f - e^{-tH_o} f}{h} \right\}}_{=\frac{\partial}{\partial t}\{e^{-tH_o} f\}} = -e^{-tH_o} H_o f\,.$$

Using Proposition 3.6 we have to $-e^{-tHo} H_o f = -H_o e^{-tHo} f$, with which it concludes.

Proposition 3.8: *If* $f \in Dom(H_o)$ therefore, the application*:* $t \to \partial\ \{e^{-tH}o\, f\} = -e^{-tHo} H_o f$*, of* $(0, \infty)$ *a* $L^2([-\pi, \pi])$, is continuous.

Test: As $f \in Dom(H_o)$ therefore, $H_o f \in L^2([-\pi, \pi])$; then using Preposition 3.3, the application is continuous.

Proposition 3.9 *Si* $f \in Dom(H_o)$ *therefore,* $e^{-(\cdot)Ho} f \in C^1((0, \infty), L^2([-\pi, \pi])$.

Test: It is a consequence of the two previous Propositions.

Another consequence is the following result.

Proposition 3.10 *The operator* $H_o : Dom(H_o) \subset L^2([-\pi, \pi]) \dashrightarrow L^2([-\pi, \pi])$ *is closed.*

Test: Since $-H_o$ is the G.I. of the contraction semigroup: $\{e^{-tHo}\}_{t\geq 0}$ $L^2([-\pi, \pi])$ we have that $-H_o$ is closed. In fact, it is: $f_k \in Dom(-H_o)$ and:

$$f_k \to f \text{ en } L^2([-\pi, \pi]) \text{ when } k \to +\infty \tag{3.16}$$

$$-H_o f_k \to g \text{ en } L^2([-\pi, \pi]) \text{ when } k \to +\infty. \tag{3.17}$$

We will prove that $f \in Dom(-H_o)$ and $-H_o f$

$= g$. From Propositions 3.7 and 3.8 we have:

$$e^{-tH_o} f_k - f_k = \int_0^t e^{-rH_o}(-H_o) f_k \, dr \, . \tag{3.18}$$

Using the continuity of e^{-tH_o} and the convergences (3.16) y (3.17) we have:

$$e^{-tH_o} f - f = \int_0^t e^{-rH_o} g \, dr \, .$$

So:

$$\frac{e^{-tH_o} f - f}{t} = \frac{1}{t}\int_0^t e^{-rH_o} g \, dr \to e^{-0H_o} g = g \, ,$$

when $t \to 0^+$.

Luego, $f \in Dom(-H_o)$ y $-H_o f = g$.

Finally, we have an important result of the solution of an initial value problem.

Proposition 3.11: The Abstract Cauchy Problem

$$(Q) \left| \begin{array}{l} u_t = -H_o u \\ u(0) = f \in Dom(H_o) \subset L^2([-\pi, \pi]) \end{array} \right.$$

has a single solution: $u(t) = e^{-tHo} f, \forall t \geq 0$, where $u \in C([0, \infty), L^2([-\pi, \pi])) \cap C^1((0, \infty), L^2([-\pi, \pi]))$.

Note 3.2: *From* Proposition 3.4, we have the solution of problem (Q)

It continually depends on the initial data.

GRAPHIC STANDARD IN DOM(H_o) $\subset L^2([-\pi, \pi])$

In order to carry out and avoid confusion with other standards that will be introduced, we will use the notation: $\| \,.\, \| L^2 := \| \,.\, \|^2$.

Definition: 3.2 *In* $Dom(H_o) \subset L^2([-\pi, \pi])$ We defined the application:

$$\begin{array}{rcl} <\cdot,\cdot>_\Delta : Dom(H_o) \times Dom(H_o) & \longrightarrow & \mathbb{C} \\ (f, g) & \longrightarrow & < f, g >_\Delta \end{array}$$

Where:

$$< f, g >_\Delta := < f, g >_{L^2} + < H_o f, H_o g >_{L^2} \, , \; \forall f, g \in Dom(H_o) \, .$$

It is observed: $< \cdot, \cdot >_\Delta$ is well defined.

Proposition 3.12: The application $< \cdot, \cdot >_\Delta$ It is an internal product in $Dom(H_o) \subset L^2([-\pi, \pi]$.

Test: It is immediate since $< \cdot, \cdot >_{L2}$ is an internal product.

Thus, the internal product $< \cdot, \cdot >_\Delta$ induces a norm

$$\|f\|_{<\cdot,\cdot>_\Delta} = \sqrt{\|f\|^2_{L^2} + \|H_o f\|^2_{L^2}} \, , \; \forall f \in Dom(H_o) \, . \tag{3.19}$$

We will denote $\| \,.\, \| < \cdot, \cdot >_{\Delta.}$ por $\| \cdot \|_\Delta$

Therefore,

Proposition 3.13 The spacenormed $(Dom(H_o), \| \cdot \|_\Delta)$ *satisfies:*

$$\|f\|_\Delta \;\geq\; \|f\|_{L^2} \, , \; \forall f \in Dom(H_o) \, , \tag{3.20}$$

$$\|f\|_\Delta \;\geq\; \|H_o f\|_{L^2} \, , \; \forall f \in Dom(H_o) \, . \tag{3.21}$$

Test: It is immediate (3.19).

Proposition 3.14 The space $(Dom(H_o), \| \cdot \|_\Delta)$ is complete.

Test: So, we have (f_n) a Cauchy sequence in $Dom(H_o)$ con $\| \cdot \|_\Delta$. We will prove: $\exists f \in Dom(H_o)$ and $\|f_n - f\|_\Delta \to 0$ when $n \to +\infty$.

It was showed: $\epsilon > 0, \exists N_o \in \mathbb{N}$ and:

$$\epsilon > \|f_n - f_m\|_\Delta \text{ whenever } n, m > N_o. \tag{3.22}$$

Of (3.20) we have:

$$\epsilon > \|f_n - f_m\|_\Delta \geq \|f_n - f_m\|_{L^2} \text{ whenever } n, m > N_o. \tag{3.23}$$

Of (3.21) we have:

$$\epsilon > \|f_n - f_m\|_\Delta \geq \|H_o(f_n - f_m)\|_{L^2} = \|H_o f_n - H_o f_m\|_{L^2} \text{ whenever } n, m > N_o. \tag{3.24}$$

From (3.23) we have that (f_n) is a Cauchy sequence in $L^2([-\pi, \pi])$, and as $L^2([-\pi, \pi])$ is complete, therefore, $\exists f \in L^2([-\pi, \pi])$ and:

$$f_n \overset{\|\cdot\|_{L^2}}{\longrightarrow} f \, . \tag{3.25}$$

From (3.24) we have that $(H_o f_n)$ is a Cauchy sequence in $L^2([-\pi, \pi])$, and as $L^2([-\pi, \pi])$ is complete, therefore, $\exists g \in L^2([-\pi, \pi])$ and:

$$H_o f_n \overset{\|\cdot\|_{L^2}}{\longrightarrow} g \, . \tag{3.26}$$

From (3.25), (3.26) and as H_o is a closed

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operator, therefore,

$$f \in Dom(H_o) \text{ y } H_o f = g\,. \tag{3.27}$$

Of (3.25), (3.26) y (3.27) we have:

$$\begin{aligned} \|f_n - f\|_\Delta^2 &= \|f_n - f\|_{L^2}^2 + \|H_o(f_n - f)\|_{L^2}^2 \\ &= \|f_n - f\|_{L^2}^2 + \|H_o f_n - H_o f\|_{L^2}^2 \to 0 \end{aligned}$$

when $n \to +\infty$.

Therefore, $\|f - f\| \to 0$ when $n \to +\infty$. Esto es, $\exists f \in Dom(H)$ tal que $f_n \overset{\|\cdot\|_\Delta}{\to} f$.

Note 3.3 *The space* $(Dom(H_o), \|\cdot\|_\Delta)$ is a Banach space or also $(Dom(H_o), <\cdot,\cdot>_\Delta)$ is a Hilbert space.

Proposition 3.15 *So, we have:*

$$\begin{aligned} H_o : (Dom(H_o), \|\cdot\|_\Delta) &\longrightarrow L^2([-\pi,\pi]) \\ f &\longrightarrow H_o f = (k^2 \widehat{f}(k))^\vee \end{aligned}$$

therefore, H_o *is a bounded operator and* $\| H_o \| \le 1$.

Test: It is immediate (3.21).

We have the following property that connects $\|\cdot\|_\Delta$ with the semigroup $\{e^{-tHo}\}_{t\geq 0}$

Proposition 3.16 *So, we have* $t \ge 0$, *si* $f^n \overset{\|\cdot\|_\Delta}{\to} f$. *therefore* $\|e^{-tH_o} f_n - e^{-tH_o} f\|_{L^2} \to 0$ *when* $n \to +\infty$.

Test: It is immediate since using (3.20) we have to $f_n \overset{\|\cdot\|_\Delta}{\to} f$ implies $f_n \overset{\|\cdot\|_{L^2}}{\to} f$, then using the Parseval Identity we have to $\widehat{f_n} \overset{\|\cdot\|_{l^2}}{\to} \widehat{f}$ and as

$$\|e^{-tH_o} f_n - e^{-tH_o} f\|_{L^2} = \|e^{-tH_o}(f_n - f)\|_{L^2} \le \sqrt{2\pi}\|\widehat{f_n} - \widehat{f}\|_{l^2}\,,$$

we conclude.

OTHERS RULES IN Dom(Ho)

Now, we will introduce other rules in $Dom(H_o)$.

Note 3.4 (p-rules in $Dom(H_o)$**) in subspace**: $Dom(H_o) \subset L^2([-\pi,\pi])$

we can define other rules, for example: $\|\cdot\|_p$, $1 \le p \le \infty$,

$$\begin{aligned} 1 \le p < \infty\,,\ \|f\|_p &:= (\|f\|_{L^2}^p + \|H_o f\|_{L^2}^p)^{\frac{1}{p}} \\ \|f\|_\infty &:= \text{máx}\{\|f\|_{L^2}, \|H_o f\|_{L^2}\} \end{aligned}$$

To: $f \in Dom(H_o)$. And it is observed that all these norms are equivalent. Note: $\| f \|_2 \, \| f \|_\Delta$

Furthermore, the following inequalities are fulfilled:

$$\|f\|_p \geq \|f\|_{L^2}\,,\ \forall f \in Dom(H_o)\,, \tag{3.28}$$

$$\|f\|_p \geq \|H_o f\|_{L^2}\,,\ \forall f \in Dom(H_o) \tag{3.29}$$

To: $p \in [1, \infty]$.

Proposition 3.17 The space$(Dom(H_o), \|\cdot\|_p)$ is complete to: $p \in [1, \infty]$.

Test: This continues since $\|\cdot\|_p$ is equivalent to $\|\cdot\|_\Delta$ and $(Dom(H_o), \|\cdot\|_\Delta)$ is complete.

Proposition 3.18 *So, we have:*

$$\begin{aligned} H_o : (Dom(H_o), \|\cdot\|_p) &\longrightarrow L^2([-\pi,\pi]) \\ f &\longrightarrow H_o f = (k^2 \widehat{f}(k))^\vee \end{aligned}$$

Therefore, H_o *is limited and* $\| H_o \| \le 1$.

Test: It is immediate from (3.29).

Besides, we have the following property that connects $\|\cdot\|_p$ with the semigroup $\{e^{-tH_o}\}_{t\geq 0}$.

Proposition 3.19 So, we have $t \ge 0$, $p \in [1, \infty]$, if $f_n \overset{\|\cdot\|_p}{\longrightarrow} f$ *so,* $\|e^{-tHo} fn - e^{-tHo} f\|_{L^2} \to 0$ *when* $n \to +1$.

Test: From (3.28) we have that $f_n \overset{\|\cdot\|_p}{\longrightarrow}$ implies $f_n \overset{\|\cdot\|_{L^2}}{\to} f$. Then, using the Parseval Identity we have to $\overset{\|\cdot\|_{l^2}}{\longrightarrow}$ Therefore

$$\|e^{-tH_o} f_n - e^{-tH_o} f\|_{L^2} =$$
$$\|e^{-tH_o}(f_n - f)\|_{L^2} \le \sqrt{2\pi}\|\widehat{f_n} - \widehat{f}\|_{l^2} \to 0$$

when $n \to +\infty$.

CLASS SEMIGROUP: C_o IN $Dom(H_o) \subset L^2([-\pi,\pi])$ WITH $\|.\|_\Delta$

Proposition 3.20 (Class Semigroup C_o **in** $Dom(H_o)$**)** *Sea* $t \ge 0$,, we defined the applications: $e^{-tH_o} f := (e^{-tk^2} \widehat{f}(k))^\vee$, $\forall f \in Dom(H_o) \subset L^2([-\pi,\pi])$ thus, $\{e^{-tH_o}\}_{t\geq 0} \subset B(Dom(H_o))$ and also forms a class contraction semigroup C_o in the space $(Dom(H_o), \|\cdot\|_\Delta)$ of Hilbert.

Test: So, we have $t > 0$ y $f \in Dom(H_o)$, using (3.11) $\{e^{-tH_o}\}_{t\geq 0}$ contraction group in $L^2([-\pi,$

π]), we have:

$$\begin{aligned} \|e^{-tH_o}f\|_\Delta^2 &= \|e^{-tH_o}f\|_{L^2}^2 + \|H_o(e^{-tH_o}f)\|_{L^2}^2 \\ &\leq \|f\|_{L^2}^2 + \|e^{-tH_o}H_of\|_{L^2}^2 \\ &\leq \|f\|_{L^2}^2 + \|H_of\|_{L^2}^2 \\ &= \|f\|_\Delta^2 \end{aligned} \quad (3.30)$$

It means,

$$\|e^{-tH_o}f\|_\Delta \leq \|f\|_\Delta\,,\ \forall f \in Dom(H_o)\,, \quad (3.31)$$

from which it follows that $e^{-tH_o} \in B(Dom(H_o))$ and $\|e^{-tH_o}\| \leq 1$.

So, we have $f \in Dom(H_o)$ and $t > 0$, using (3.11) and (3.6) we have

$$\begin{aligned} &\|e^{-tH_o}f - f\|_\Delta^2 \\ = &\|e^{-tH_o}f - f\|_{L^2}^2 + \|H_o(e^{-tH_o}f - f)\|_{L^2}^2 \\ = &\|e^{-tH_o}f - f\|_{L^2}^2 + \|e^{-tH_o}H_of - H_of\|_{L^2}^2 \to 0 \end{aligned} \quad (3.32)$$

when $t \to 0^+$.

Since (3.1) and (3.4) are also satisfied, we conclude that $\{e^{-tH_o}\}_{t\geq 0}$ is a class contraction semigroup C_o in $Dom(H_o)$.

Proposition 3.21 So, we have $f \in Dom(H_o)$, the application: $t \to e^{-tH_o} f$ *is* continuous $[0, \infty)$ *a* $Dom(H_o)$.

Test: so, we have $f \in Dom(H_o)$ and $t > 0$, Using (3.11) and Proposition 3.3, we have:

$$\begin{aligned} &\|e^{-(t+h)H_o}f - e^{-tH_o}f\|_\Delta^2 \\ = &\|e^{-(t+h)H_o}f - e^{-tH_o}f\|_{L^2}^2 + \|H_o(e^{-(t+h)H_o}f - e^{-tH_o}f)\|_{L^2}^2 \\ = &\|e^{-(t+h)H_o}f - e^{-tH_o}f\|_{L^2}^2 + \|e^{-(t+h)H_o}H_of - e^{-tH_o}H_of\|_{L^2}^2 \to 0 \end{aligned}$$

when $h \to$ o.

Proposition 3.22 *is* $t \geq 0$, *si* $f_n \overset{\|\cdot\|_\Delta}{\longrightarrow}$ thus, $e^{-tH_o} f_n \overset{\|\cdot\|_\Delta}{\longrightarrow} e^{-tH_o} f$.

Test: Using (3.31) con $f_n - f \in Dom(H_o)$, we have

$$\|e^{-tH_o}f_n - e^{-tH_o}f\|_\Delta =$$

$$\|e^{-tH_o}(f_n - f)\|_\Delta \leq \|f_n - f\|_\Delta \to 0$$

when $n \to +\infty$.

EXISTENCE OF SOLUTION

Thus, from Propositions 3.11 and 3.21, we obtain the following result for the existence of a solution.

Proposition 3.23: it *is* $t \geq 0$, $e^{-tH_o} f = (e^{-tk^2} \hat{f}(k))^{\vee}$, $\forall f \in L^2([-\pi, \pi])$. Then the Abstract Cauchy Problem

$$(Q)\ \left|\ \begin{array}{l} u_t = -H_o u \\ u(0) = f \in Dom(H_o) \subset L^2([-\pi,\pi]) \end{array}\right.$$

has a single solution*:* $u(t) = e^{-tH_o} f$, $\forall t \geq 0$, with $u \in C([0, \infty), Dom(H_o)) \cap C^1((0, \infty), L^2([-\pi, \pi]))$, where we consider $Dom(H_o)$ with the graph norm $\|\cdot\|_\Delta$.

Note 3.5: *I*n proposition 3.23, we can consider $\|\cdot\|_p$ instead of the norm of the graph, since they are equivalent.

Note 3.6: The continuous dependence of the solution with respect to the initial data is obtained, in the versions: Propositions 3.4, 3.16, 3.19 and 3.22.

CONCLUSIONS

In our study, we have done the following:

1. We remember the differential operator: H_o in $L^2([-\pi, \pi])$, which is densely defined, unbounded, symmetric and does not admit simple linear extensionetrica a $L^2([-\pi, \pi])$.

2. We introduced a family of operators and prove that it forms a contraction semigroup of class C_o over $L^2([-\pi, \pi])$.

3. We showed that $-H_o$ is the infinitesimal generator of said contraction semigroup on $L^2([-\pi, \pi])$. And it is also obtained that the associated Abstract Cauchy Problem (PCA) is well placed.

4. We introduced a norm in the domain of H_o: $Dom(H_o) \subset L^2([-\pi, \pi])$, what makes H_o is limited, and we introduce other norms equivalent to this.

5. We proved that the restrictions on

Dom(H_o) of the semigroup operators: C_o *about space:* $L^2([-\pi, \pi])$, they also form a semigroup C_o *over space Dom* (H_o) of Hilbert.

6. We obtained a better solution existence result from the associated PCA.

7. The properties obtained can be generalized to periodic Sobolev spaces and therefore applied in the study of the existence of solutions to evolution equations

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